International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 4, August2017, ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
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SOME FIXED POINT AND COMMON FIXED POINT THEOREMS ON ORDERED CONE METRIC SPACES

Vishnu Bairagi[1],V.H.Badshah[2] and Aklesh Pariya[3]

1. Department of Mathematics, Govt. M. L. B. Girls, P. G. College, Indore, (M.P), India

2. School of Studies in Mathematics, Vikram University, Ujjain, (M.P.), India

3. Department of Mathematics, Medi-Caps University, Indore, (M.P), India

Abstract: *The purpose of this paper is to obtain, fixed point and common fixed point theorems for self-maps on ordered cone metric space, satisfying certain contractive conditions on partially ordered cone metric space, which shows that our main results are more useful than the presented results in some recent literatures*

Keywords:*Fixed point theorem, Cone metric space, Ordered set, Normal cone*

Mathematics Subject Classification (2000) 47H10, 54H25

1.0 Introduction: Cone metric spaces were introduced by Huang and Zhang[7].In 2008, few fixed point results of some mapping with certain contractive property on cone metric spaces have been established by Abbas andJungck[1]. In 2008 by Ilic and Rakocevic [8] and by Abbas and Rhoades [2]established some fixed point results on cone metric spaces.Huang and Zhang[7], investigated the convergence of a sequence in cone metric spaces in order to introduce the notionof completeness and proved some fixed point theorems for contractive maps on partially ordered cone metric space. Existence of fixed point in partially ordered sets has been considered recently by Ran andReuring[15].In the past several years some existence results of fixed points for some contractive type maps in partially ordered cone metric spaces were investigated.In 2009,Altun and Durmaz[20] established some fixed point theorems on ordered cone metric space and in 2010,Altun,et al.[21] established fixed point and common fixed point theorems on ordered cone metric spaces. Our result generalizes and improves the result of fixed point theorems established by Altun, et al.[21].

Definition 1.1 [7]Let Ebe a real Banach space and P be a subset of E. Letθ denotes the zero element of E and by $IntP$ the interior of P. The subset P is called a cone if and only if:

(i)P is closed, non-empty and $P \neq \{\theta\}$,

(ii)$a,b \in \mathbb{R}$, $a,b \geq 0$, $x, y \in P \Rightarrow ax+by \in P$,

(iii)$x \in P$and$- x \in P \Rightarrow x = \{\theta\}$.

A cone P is called solid if it contains interior points that is if $Int\ P \neq \phi$

Given a cone $P \subset E$, we define a partial ordering $\preccurlyeq$ with respect to P, by$x \preccurlyeq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preccurlyeq y$ and $x \neq y$, we shall write $x \ll y$ if $y - x \in Int\ P$.

The cone P in a real Banach Space $E$is called normal if there is a number $K\geq 1$ such that for all $x, y \in E$, $\theta \preccurlyeq x \preccurlyeq y$ implies that $\|x\| \preccurlyeq K\|y\|$**… (1)**

Theleast positivenumber$K$ satisfyingthe (1) is called thenormal constant of P.

In the following we always suppose that E is a Banach space, P is a cone in E with $Int\ P \neq \phi$and $\preccurlyeq$ is partial ordering with respect to P.

Definition 1.2 [7] Let X be a non-empty set and let $d : X\times X\rightarrow E$bea mapping satisfies the following conditions

(i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x = y$ and $d(x, y) = \theta$ if and only if $x = y$,

(ii) $d(x, y) = d(y,x)$ for all $x, y \in X$,

(iii) $d(x, y) \preccurlyeq d(x,z)+d(z, y)$ for all $x, y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

***Example* 1. [7]**Let $E = \mathrm{R}^2$, $P =\{(x, y) \in E \mid x, y \geq 0\}$, $X = \mathrm{R}$ and $d : X \times X \rightarrow E$ such that $d(x, y) = \{\frac{1}{2}|x-y|, \frac{\alpha}{2}|x-y|\}$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 1.3 [7] Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in$X$ and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is an N such that for all $n > N$, $d(x_n,x) \ll c$,then $\{x_n\}$ is saidto beconvergent and$\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denotethis by $\lim_{n\rightarrow\infty} x_n = x$ or$x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 1.4 [7]Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in$X$ and $x \in X$.If for every $c \in E$ with $\theta \ll c$ there is an N such that for all $n,m > N$, $d(x_n,x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X. A cone metric space(X,d) iscalled complete cone metric space if every Cauchy sequence in $(X,d)$is convergentin X.

Definition 1.5[7]Let $(X,\sqsubseteq)$ be a partially ordered set. Two mappings $f, g:X \rightarrow X$ are said to be weakly increasing if $fx\sqsubseteq gfx$ and $gx\sqsubseteq fgx$are hold for all $x \in X$.

Theorem 1.1 ([11],[15]) Let$(X,\sqsubseteq)$ be a partiallyordered set and supposethat there exists a metric d in X such that the metric space(X,d) is complete. Let$f : X \rightarrow X$ be a continuousand non-decreasingmappingw.r.t. $\sqsubseteq$Supposethatthe following two assertions hold:

(i)there exists $k \in (0,1)$such that $d(fx, fy) \leq kd(x, y)$ for each $x, y \in X$ with $y \sqsubseteq x$;

(ii)thereexists $x_0 \in X$ such that $x_0 \sqsubseteq f x_0$.

Then f has a fixed point $x^* \in X$.

Some generalizations and variants of the result of [15] are given in [3], [11],[12] and [13].

For example,in [11], the following theorem has been proved by removing the continuity of f in theorem 1.1

Theorem 1.2 [11]Let $(X, \sqsubseteq)$bea partiallyordered set and supposethat there exists a metric d in X such that the metric space $(X,d)$is complete. Let $f : X \rightarrow X$ be a non-decreasing mapping w.r.t$\sqsubseteq$. Suppose that the followingthree assertions hold:

(i) there exists $k \in(0,1)$ such that $d(f x, f y) \leq kd(x, y)$ for each $x, y \in X$ with $y \sqsubseteq x$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;

(iii)if an increasing sequence$\{x_n\}$ converges to x in X, then$x_n \sqsubseteq x$ for all n.

Then f has a fixed point $x^* \in X$.

In Theorem 1.1it is proved that, if every pair of elements has a lower bound and an upper bound,then for every $x \in X$,$\lim_{n\to\infty} f^n x = y$, where$y$ is the fixed point of $f$such that

$y = \lim_{n\to\infty} f^n$and hence f has a uniquefixed point.

Lemma 1.[7]Let (X,d)be a cone metric space, P be a normal cone and let$\{x_n\}$ be a sequencein X. Then

(i) $\{x_n\}$ converges to x if and only if $d(x_n, x) \rightarrow \theta$ *as* $n \rightarrow \infty$. ,

(ii) $\{x_n\}$ *is a Cauchy sequence if and only if* $d(x_n,x_m) \rightarrow \theta$ as, $m\ n \rightarrow \infty$..

Let (X,d)beaconemetricspace, $f : X \rightarrow X$ and $x_0 \in X$.Then thefunction f is continuous at x_0 if for any sequence $x_n \rightarrow x_0$ we have $f x_n \rightarrow f x_0$.

Theorem 1.3 [7]Let (X, d)be a complete cone metric space, P be a normal cone with normal constant k suppose the mapping $f : X \rightarrow X$satisfies the contractive condition

$d(f x, f y) \leq kd(x, y)$ for each $x, y \in X$, *where* $k \in [0,1)$ is constant.

Then f has a fixed pointin X.

In 2009, Altun and Durmaz[20] have proved the following theorem [1.4] by using ideas of theorems 1.1 and 1.3

Theorem 1. 4[20]Let$(X, \sqsubseteq)$ be a partiallyordered set and supposethat there exists a cone metric d in X such that the cone metric space(X,d) is complete and$P$be a normal cone with normal constant$K$. Let $f : X \rightarrow X$ be a continuous andnon-decreasingmappingw.r.t.$\sqsubseteq$ .Supposethatthefollowingtwo assertions hold:

(i)there exists $k \in(0,1)$ such that $d(f x, f y) \leq kd(x, y)$ for each $x, y \in X$ with $y \sqsubseteq x$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq f x_0$.

Then f has a fixed point $x^* \in X$.

In 2010,Altun,I. Damjanovic, B. and Djoric, D. given a generalized version of theorem 1.4 in the ordered cone metric spaces, where a cone P is not necessarily normal.

Theorem 1. 5[21]Let$(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space $(X,d)$is complete.Let$f : X \to X$ be a continuous and non-decreasing mapping w.r.t. $\sqsubseteq$. Suppose that the following two assertions hold:

(i)there exists$\alpha, \beta, \gamma \geq 0$with $\alpha + 2\beta + 2\gamma < 1$ such that

$(fx, fy) \preccurlyeq \alpha\, d(x,y) + \beta\, [d(x, fx)+d(y, fy)] + \gamma[d(x, fy) + d(y, fx)]$...**(2)**

or each $x, y \in X$ with$y \sqsubseteq x$;

(ii)there exists $x_0 \in X$ such that $x_0 \sqsubseteq f\, x_0$.

Then f has a fixed point $x^* \in X$.

The purpose of my work is to present generalized version of theorem 1.5 [21] for ordered cone metric space.

2. Main results:

Theorem 2.1 Let$(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a conemetric d in X such that the cone metric space$(X,d)$is complete. *Let* $f : X \to X$ be a continuous and non-decreasing mapping w.r.t.$\sqsubseteq$. Suppose that the following two assertions hold:

(i)there exists$\alpha, \beta, \gamma, \eta \geq 0$with $3\alpha + 2\beta + 2\gamma + \eta < 1$such that

$$d(fx, fy) \preccurlyeq \alpha\, [d(x, y)+ d(y, fx)] + \beta\, [d(x, fx)+ d(y, fy)] + \gamma\, [d(x, fy) + d(y, fx)]$$

$$+\eta\, d(y, fy)$$ for each $x, y \in X$ with $y \sqsubseteq x$;...**(3**)

(ii)there exists $x_0 \in X$ such that $x_0 \sqsubseteq f\, x_0$.

Then f has a fixed point $x^* \in X$.

Proof. Let $f\, x_0 = x_0$, then the proof is completed. Suppose that $f\, x_0 \neq x_0$. Since $x_0 \preccurlyeq f\, x_0$ and f is non-decreasing w.r.t. $\preccurlyeq$, we obtain by induction that

$$x_0 \sqsubseteq f\, x_0 \sqsubseteq f^2 x_0 \sqsubseteq \cdots \sqsubseteq f^n x_0 \sqsubseteq f^{n+1} x_0 \sqsubseteq \ldots.$$

Now, we have

$$d(f^{n+1}x_0, f^n x_0)=d(f (f^n x_0), f (f^{n-1}x_0))$$

$$\preccurlyeq \alpha[d(f^n x_0,f^{n-1}x_0)+d(f^{n-1}x_0,f^{n+1}x_0)] + \beta\, [d(f^n x_0, f^{n+1}x_0)+d(f^{n-1}x_0, f^n x_0)]$$

$$+ \gamma[d(f^n x_0, f^n x_0) + d(f^{n-1}x_0, f^{n+1}x_0)] +\eta\, d(f^{n-1}x_0, f^n x_0)$$

$$\preccurlyeq \alpha[d(f^n x_0,f^{n-1}x_0)+d(f^{n-1}x_0,f^n x_0)+d(f^n x_0,f^{n+1}x_0)]$$

$$\beta\, [d(f^n x_0, f^{n+1}x_0)+d(f^{n-1}x_0, f^n x_0)]+ \gamma[d(f^{n-1}x_0, f^n x_0)+d(f^n x_0, f^{n+1}x_0)]$$

$$+\eta\, d(f^{n-1}x_0, f^n x_0)$$

$$(2\alpha+\beta+\gamma+\eta)d(f^n x_0,f^{n-1}x_0)+ (\alpha+\beta+\gamma)d(f^n x_0,f^{n+1}x_0)$$

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Vol. 6 Issue 4, August 2017, ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
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and so

$(f^{n+1}x_0, f^n x_0) \preccurlyeq \frac{(2\alpha+\beta+\gamma+\eta)}{(1-\alpha-\beta-\gamma)} d(f^n x_0, f^{n-1}x_0)$

$(f^{n+1}x_0, f^n x_0) \preccurlyeq kd(f^n x_0, f^{n-1}x_0)$ for all $n \geq 1$,

here $k = \frac{(2\alpha+\beta+\gamma+\eta)}{(1-\alpha-\beta-\gamma)}$

epeating this relation up to n times we get

$(f^{n+1}x_0, f^n x_0) \preccurlyeq k^n d(f^n x_0, f^{n-1}x_0)$**…(4)**

Let $m > n$ then from(4)

$(f^m x_0, f^n x_0) \preccurlyeq d(f^m x_0, f^{m-1}x_0) + d(f^{m-1}x_0, f^{m-2}x_0) + \ldots + d(f^{n-1}x_0, f^n x_0)$

$(k^{m-1} + k^{m-2} + \ldots + k^n) d(f x_0, x_0)$

$\preccurlyeq (k^n + k^{n-1} + \ldots + k^{m-2} + k^{m-1}) d(f x_0, x_0)$

$\preccurlyeq \frac{k^n}{1-k} d(f x_0, x_0)$

Therefore we get

$d(f^m x_0, f^n x_0) \preccurlyeq \frac{k^n}{1-k} d(f x_0, x_0)$ **… (5)**

Now we show that $\{f^n x_0\}$ is a Cauchy sequence in (X, d). Let $\theta \ll c$ be arbitrary.

Since $c \in Int\ P$, there is a neighbourhood of θ :

$N_\delta(\theta) = \{y \in E : \|y\| < \delta\}$, $\delta > 0$, such that $c + N_\delta(\theta) \subseteq Int\ P$. Choose a natural number N_1 such that $\left\| -\frac{k^{N1}}{1-k} d(f x_0, x_0) \right\| < \delta$. Then $-\frac{k^{N1}}{1-k} d(f x_0, x_0) \in N_\delta(\theta)$ for all $n \geq N_1$.

Hence $c - \frac{k^n}{1-k} d(f x_0, x_0) \in c + N_\delta(\theta) \subseteq Int\ P$.

Thus we have

$\frac{k^n}{1-k} d(f x_0, x_0) \ll c$ for all $n \geq N_1$.

Therefore, from (5)we get,

$d(f^m x_0, f^n x_0) \preccurlyeq \frac{k^n}{1-k} d(f x_0, x_0) \ll c$ for all $m > n \geq N_1$, and hence

$d(f^m x_0, f^n x_0) \ll c$ for all $m > n \geq N_1$.

ence we conclude that, $\{f^n x_0\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone metric space, there exists $x^* \in X$ such that $f^n x_0 \to x^*$ as $n \to \infty$.

Finally, continuity of f and $f(f^n x_0) = f^{n+1}x_0 \to x^*$ imply that $fx^* = x^*$.

Thus we proved that $x^*$is a fixed point of f.

If we use the condition (iii) instead of the continuity of f in theorem 2.1 we have the following result.

Theorem 2.2 Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete.Let$f : X \to X$ be a non-decreasing mapping $w.r.t.\sqsubseteq$. Suppose that the following three assertions hold:

(i)there exist $\alpha,\beta,\gamma,\eta \geq 0$ with$3\alpha + 2\beta + 2\gamma + \eta < 1$ such that

$$d(fx, fy) \preccurlyeq \alpha [d(x, y) + d(y, fx)] + \beta [d(x, fx) + d(y, fy)] + \gamma [d(x, fy) + d(y, fx)]$$

$$+\eta\, d(y, fy) \text{ for all } x, y \in X \text{ with } y \sqsubseteq x, \quad \ldots\mathbf{(6)}$$

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$,

(iii) If an increasing sequence$\{x_n\}$converges to x in X, then$x_n \sqsubseteq x$ for all n.

Then f has a fixed point $x^* \in X$.

Proof. If we take $x_n = f^n x_0$ in the proof of theorem 2.2, then we have $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$, that is, $\{x_n\}$ is an increasing sequence. Also this sequence converges to x^*. Now the condition (iii) implies $x_n \sqsubseteq x^*$for all n. Therefore, we can use the condition (i) and so we have

$$d(fx_n, fx^*) \preccurlyeq \alpha [d(x_n, x^*) + d(x^*, fx_n)] + \beta [d(x_n, fx_n) + d(x^*, fx^*)] + \gamma [d(x_n, fx^*) + d(x^*, fx_n)]$$

$$+\eta\, d(x^*, fx^*)$$

Taking $n \to \infty$, we have

$$d(x^*, fx^*) \preccurlyeq (\beta + \gamma + \eta) d(x^*, fx^*).$$

Then by the condition (i) we have $d(x^*, fx^*) \preccurlyeq \frac{1}{2} d(x^*, fx^*)$ and hence $\frac{1}{2} d(x^*, fx^*) \preccurlyeq \theta$

Therefore, $-d(x^*, fx^*) \in P$ and so, as also $d(x^*, fx^*) \in P$, we have $d(x^*, fx^*) = \theta$.

Hence $fx^* = x^*$.

Nowwe give two common fixed point theorem on ordered cone metric spaces.

Theorem 2.3 Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let$f, g : X \to X$ be two weakly increasing mappings $w.r.t. \sqsubseteq$. Suppose that the following two assertions hold:

(i) there exist $\alpha,\beta,\gamma,\eta \geq 0$ with$\alpha + 2\beta + 2\gamma + \eta < 1$ such that

$$d(fx, gy) \preccurlyeq \alpha [d(x, y) + d(y, fx)] + \beta [d(x, fx) + d(y, gy)] + \gamma [d(x, gy) + d(y, fx)]$$

$$+\eta\, d(y, gy) \text{ for all comparative } x, y \in X, \quad \ldots\mathbf{(7)}$$

(ii) $f$or g is continuous.

Then f and g have common fixed point $x^*\in X$.

Proof. Let x_0 be an arbitrary point of X and define a sequence $\{x_n\}$ in X such that, $x_{2n+1}= fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \geq 0$. Note that, since f and g are weakly increasing, we have $x_1 = fx_0 \sqsubseteq gfx_0 = gx_1 = x_2$, and $x_2 = gx_1 \sqsubseteq fgx_1 = fx_2 = x_3$ and continuing this process we have $x_1 \sqsubseteq x_2 \sqsubseteq \ldots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \ldots$ that is, the sequence $\{x_n\}$ is non-decreasing. Now, since x_{2n} and x_{2n+1} are comparative, we can use the inequality (7)and then we have

$$d(x_{2n+1}, x_{2n+2}) = d(fx_{2n}, gx_{2n+1})$$

$$\preccurlyeq \alpha\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, fx_{2n})] + \beta\, [d(x_{2n}, fx_{2n}) + d(x_{2n+1}, g\, x_{2n+1})]$$

$$+ \gamma\, [d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})] + \eta\, d(x_{2n+1}, gx_{2n+1})$$

$$\preccurlyeq \alpha\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+1})] + \beta\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+ \gamma\, [d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})] + \eta\, d(x_{2n+1}, x_{2n+2})$$

$$\preccurlyeq \alpha\, [d(x_{2n}, x_{2n+1})] + \beta\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+ \gamma\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \eta\, d(x_{2n+1}, x_{2n+2})$$

$$\preccurlyeq (\alpha + \beta + \gamma) d(x_{2n}, x_{2n+1}) + (\beta + \gamma + \eta) d(x_{2n+1}, x_{2n+2})$$

Which implies that

$$d(x_{2n+1}, x_{2n+2}) \preccurlyeq \frac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma-\eta)} d(x_{2n}, x_{2n+1})$$

$$d(x_{2n+1}, x_{2n+2}) \preccurlyeq k\, d(x_{2n}, x_{2n+1})$$

where $k = \frac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma-\eta)} < 1$. Similarly, it can be shown that $d(x_{2n+3}, x_{2n+2}) \preccurlyeq k d(x_{2n+2}, x_{2n+1})$

Therefore,

$d(x_{n+1}, x_{n+2}) \preccurlyeq k d(x_n, x_{n+1}) \preccurlyeq k^2 d(x_{n-1}, x_n) \preccurlyeq \ldots \preccurlyeq k^{n+1} d(x_0, x_1)$ for all $n \geq 1$.

Let $m > n$; then we have

$$d(x_m, x_n) \preccurlyeq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)$$
$$\preccurlyeq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \ldots + k^{m-1} d(x_0, x_1)$$
$$\preccurlyeq [\, k^n + k^{n+1} + \ldots + k^{m-1}\,] d(x_0, x_1)$$
$$\preccurlyeq k^n [1 + k^1 + k^2 \ldots + k^{m-n-1}\,]\, d(x_0, x_1)$$

Therefore, we get

$$d(x_m, x_n) \preccurlyeq \frac{k^n}{1-k} d(x_0, x_1).$$

Hence, like in the proof of Theorem 2.1, one can prove that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Suppose that f is continuous, then it is clear that x^* is a fixed point of f.

Now we show that x^* is also a fixed point of g.

Since $x^* \sqsubseteq x^*$, we can use then equality (7) for $x = y = x^*$, then we have

$$d(f x^*, g x^*) \preccurlyeq \alpha\, [d(x^*, x^*) + d(x^*, f x^*)] + \beta\, [d(x^*, f x^*) + d(x^*, g x^*)]$$

$$+ \gamma\, [d(x^*, g x^*) + d(x^*, f x^*)] + \eta\, d(x^*, g x^*) \ldots \textbf{(8)}$$

i.e. $d(x^*, g x^*) \preccurlyeq (\beta + \gamma + \eta)\, d(x^*, g x^*)$

Then by condition $x \preccurlyeq y \Leftrightarrow y - x \in P \ \forall\, x, y \in P$ we have,

$(\beta + \gamma + \eta) d(x^*, g x^*) - d(x^*, g x^*) \in P$ i.e. $-(1-(\beta + \gamma + \eta)) d(x^*, g x^*) \in P$

Hence $(1 - (\beta + \gamma + \eta))^{-1} > 0$.

Now by the condition $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax+by \in P$

$(1 - (\beta + \gamma + \eta))^{-1} > 0$ & $-(1-(\beta + \gamma + \eta) d(x^*, gx^*) \in P$

$\Rightarrow -(1 - (\beta + \gamma + \eta))^{-1} (1-(\beta + \gamma + \eta) d(x^*, gx^*) \in P$

$\Rightarrow -d(x^*, gx^*) \in P$

Also we have $d(x^*, gx^*) \in P$

i.e $d(x^*, gx^*) \in P$. & $-d(fx^*, gx^*) \in P. \Rightarrow d(x^*, gx^*) = \{\theta\}$ and hence $gx^* = x^*$.

Similarly if g is continuous, we have $fx^* = x^*$.hece we have $gx^* = x^* = fx^*$.

Therefore f and g have common fixed point x^*.

Theorem 2.4 Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. *Let* $f, g : X \to X$ be two weakly increasing mappings *w.r.t.* $\sqsubseteq$. Suppose that the following two assertions hold:

(i) there exist $\alpha, \beta, \gamma, \eta \geq 0$ with $\alpha + 2\beta + 2\gamma + \eta < 1$ such that

$$d(fx, gy) \preccurlyeq \alpha\, [d(x, y) + d(y, fx)] + \beta\, [d(x, fx) + d(y, gy)] + \gamma\, [d(x, gy) + d(y, fx)]$$

$$+ \eta\, d(y, gy) \text{ for all comparative } x, y \in X, \ldots \textbf{(9)}$$

(ii) if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all n.

Then f and g have a common fixed point $x^* \in X$.

Proof of this theorem can be established, similar as established in theorem 2.2.

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